

SUP NORMS OF CAUCHY DATA OF EIGENFUNCTIONS ON MANIFOLDS WITH CONCAVE BOUNDARY

CHRISTOPHER D. SOGGE AND STEVE ZELDITCH

ABSTRACT. We prove that the Cauchy data of Dirichlet or Neumann Δ - eigenfunctions of Riemannian manifolds with concave (diffractive) boundary can only achieve maximal sup norm bounds if there exists a self-focal point on the boundary, i.e. a point at which a positive measure of geodesics leaving the point return to the point. In the case of real analytic Riemannian manifolds with real analytic boundary, maximal sup norm bounds on boundary traces of eigenfunctions can only be achieved if there exists a point on the boundary at which all geodesics loop back. As an application, the Dirichlet or Neumann eigenfunctions of Riemannian manifolds with concave boundary and non-positive curvature never have eigenfunctions whose boundary traces achieve maximal sup norm bounds.

1. INTRODUCTION

Let (X, g) be a compact Riemannian manifold of dimension n , and let $M = X \setminus \mathcal{O}$ be the exterior of a finite disjoint union of open convex subdomains ('obstacles'). Thus, (M, g) is a Riemannian manifold with geodesically *concave* boundary ∂M . We consider the eigenvalue problem,

$$\begin{cases} -\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda, \\ B\varphi_\lambda = 0 \text{ on } \partial M \end{cases},$$

where B is the boundary operator, either $B\varphi = \varphi|_{\partial M}$ in the Dirichlet case or $B\varphi = \partial_\nu \varphi|_{\partial M}$ in the Neumann case. We denote by $\{\varphi_j\}$ an orthonormal basis of eigenfunctions, $\langle \varphi_j, \varphi_k \rangle = \delta_{jk}$, with $\lambda_1 < \lambda_2 \leq \dots$ counted with multiplicity. The inner product is defined by $\langle f, g \rangle = \int_M f \bar{g} dA$ where dA is the area form of g .

This article is concerned with sup norm bounds on the Cauchy data

$$(\varphi_j|_{\partial M}, \lambda_j^{-1} \partial_\nu \varphi_j|_{\partial M})$$

of Dirichlet (resp. Neumann) eigenfunctions along the boundary. The non-trivial component of the Cauchy data of eigenfunctions satisfying boundary conditions is often called the 'boundary trace'. We denote by $r_q u$ the restriction of $u \in C(\bar{M})$ to ∂M at the point $q \in \partial M$, and we denote by γ_q^B the boundary trace with boundary conditions B . Thus, $\gamma_q^B = r_q$ in the

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Neumann case and $\gamma_q^B = r_q \partial_{\nu_q}$ in the Dirichlet case. We also denote by

$$\varphi_j^b(q) = \gamma_q^B \varphi_j \quad (1.1)$$

the non-trivial boundary trace of an eigenfunction satisfying the boundary condition B (either Dirichlet or Neumann).

The goal of this article, as in [SZ02, STZ11, SZ] is first to prove universal sup norm bounds on boundary traces of eigenfunctions and more significantly to characterize the Riemannian manifolds with concave boundary where the sup norm bounds are achieved. When the universal bounds are achieved, we say that $(M, g, \partial M)$ has *maximal eigenfunction growth along the boundary*. The sup norm problem on Cauchy data makes sense for any Riemannian manifold with boundary, but we restrict to the case of concave boundaries in this article. Further motivation to study manifolds with concave boundary is that the sup norm results are needed in [JZ] to count nodal domains on non-positively curved surfaces with concave boundary.

Before stating the results, we recall what is known about global sup norm bounds when $\partial M = \emptyset$ and also in the interior of M when $\partial M \neq \emptyset$. The ‘universal’ sup norm bound

$$\|\varphi_j\|_{L^\infty} \leq C_g \lambda_j^{\frac{n-1}{2}} \quad (1.2)$$

on n -dimensional Riemannian manifolds without boundary was extended to manifolds with boundary in [Gr, Sm, Sog02]. But this bound is rarely achieved, and in the articles [SZ02, STZ11, SZ] there are successively stronger constraints on Riemannian manifolds without boundary for which the bound (1.2) is achieved. We refer to such Riemannian manifolds ‘ (M, g) as having maximum eigenfunction growth’, and express the condition as

$$\|\varphi_j\|_{L^\infty} = \Omega(\lambda_j^{\frac{n-1}{2}}), \quad (1.3)$$

where Ω is the negation of “little oh”, i.e. the Ω -symbol indicates that there exists some subsequence of eigenfunctions φ_j with $\lambda_j \rightarrow \infty$ such that the standard bound is achieved. In the boundary-less case of [SZ02], it is proved that a necessary condition for maximal growth (1.3) is that there exists a ‘self-focal point’ or ‘partial blow down point’ p . We will define the terms below. The result was improved in [STZ11] and further improved in [SZ], but we will only be concerned with the original condition in this article.

The first result of this article gives universal sup norm bounds on Cauchy data parallel to those of [Gr, Sm, Sog02] in the interior. We denote by

$$\Pi_{[0, \lambda]}^b(q, q') = \sum_{j: \lambda_j \leq \lambda} \varphi_j^b(q) \varphi_j^b(q') \quad (1.4)$$

the boundary trace of the spectral projection for Δ for the interval $[0, \lambda]$.

PROPOSITION 1. *For any C^∞ Riemannian manifold (M, g) with C^∞ concave boundary ∂M ,*

$$\Pi_{[0, \lambda]}^b(q, q) = \begin{cases} C\lambda^n + R_D^b(\lambda, q), & \text{Dirichlet} \\ C\lambda^n + R_N^b(\lambda, q), & \text{Neumann.} \end{cases}$$

with

$$R_B^b(\lambda, q) = O(\lambda^{n-1}) \text{ uniformly in } q.$$

Universal sup norm bounds on boundary traces of eigenfunctions are obtained from the jump in the remainder:

COROLLARY 2. *Under the assumptions above,*

$$\sum_{j: \lambda_j = \lambda} |\varphi_j^b(q)|^2 = R_B^b(\lambda, q) - R_B^b(\lambda - 0, q) = O(\lambda^{n-1}). \quad (1.5)$$

The result was stated but not proved in [Z4]. It is proved in this article for the case of concave boundary. We also need the following generalized pointwise Weyl law for Cauchy data of eigenfunctions. We denote by $Op_h(a)$ a semi-classical pseudo-differential operator on ∂M . We refer to [Zwo12, HZ04, TZ13] for background.

PROPOSITION 3. *Let (M, g) be a compact Riemannian manifold with concave boundary. There is a constant C depending only on (M, g) so that if $Op_h(a)$ is a semi-classical zero order pseudodifferential operator on ∂M with principal symbol $a_0(q, \eta)$ vanishing in an ε -neighborhood of the glancing set, then*

$$\sum_{\lambda_j \in [\lambda, \lambda+1]} |Op_h(a)\varphi_j^b(y)|^2 = C\lambda^{n-1} \int_{B^*\partial M} |a_0(y, \xi)|^2 d\xi + O_A(\lambda^{n-2}). \quad (1.6)$$

On the other hand, if the principal symbol is supported in an ε -neighborhood neighborhood of the glancing set, then

$$\sum_{\lambda_j \in [\lambda, \lambda+1]} |Op_h(a)\varphi_j^b(y)|^2 \leq C\|a_0\|_{L^\infty} \varepsilon \lambda^{n-1} + O_a(\lambda^{n-2}) \quad (1.7)$$

We refer to the universal bounds as ‘convexity bounds’, and improvements in the case of special geometries as ‘sub-convexity bounds’. Our goal is to characterize geometries of $(M, g, \partial M)$ for which sub-convexity holds. To state our main result we now introduce the relevant geometric features.

1.1. Loops and self-focal points. We denote by G^t the billiard flow (or broken geodesic flow) of $(M, g, \partial M)$. We also denote the broken exponential map by $\exp_x \xi = \pi G^1(x, \xi)$. We refer to [Hör90] (Chapter XXIV) or [MT] for background on these notions.

Given any $x \in \bar{M}$, we denote by \mathcal{L}_x the set of loop directions at x ,

$$\mathcal{L}_x = \{\xi \in S_x^*M : \exists T : \exp_x T\xi = x\}. \quad (1.8)$$

Definition: We say that x is a self-focal point if $|\mathcal{L}_x| > 0$ where $|\cdot|_z$ denotes the surface measure on S_x^*M determined by the metric g_x .

The main result of [SZ02] is that if (M, g) is C^∞ , $\partial M = \emptyset$, and has maximal sup-norm growth in the sense of (1.3), then it must possess at least one point p for which $|\mathcal{L}_p| > 0$, i.e. a self-focal point. The main result of this article proves the same result for Cauchy data of manifolds with concave boundary.

Theorem 1.1. *Let (M, g) be a Riemannian manifold of dimension n with geodesically concave boundary. Suppose that there exist no self-focal points $q \in \partial M$. Then the sup-norms of Cauchy data of eigenfunctions are $o(\lambda_j^{\frac{n-1}{2}})$.*

Thus, the Cauchy data can only achieve maximal sup norm bounds if there exists a self-focal point on the boundary. The proof relies on three results on boundary traces, some of which are valid without the concavity assumption. The first (and simplest) result consists of upper bounds on the wave front set of the boundary trace of the wave kernel (§2, (2.5) and (2.6).) The second result consists of the pointwise Weyl laws with remainder estimates for Cauchy data of eigenfunctions in Propositions 1 and Proposition 3. These propositions are the most substantial new feature of the proof and the ones which make use of the concavity assumption. In §4 we prove Proposition 1 and in §4.4 we prove Proposition 3. The proof of Theorem 1.1 is given in §5, following the strategy in [SZ02].

We assume concavity of the boundary in the pointwise Weyl laws in order to calculate the singularity at $t = 0$. When the boundary is concave, the Cauchy data $E_B^b(t, q, q')$ of the cosine wave kernel $E_B(t, x, y)$ with boundary condition B (Dirichlet or Neumann) has a singularity only at $t = 0$ for sufficiently small $|t|$, since geodesics of the exterior of a convex obstacle cannot travel from q to q' on ∂M for small times. The calculation uses the parametrix for the wave group in the case of concave boundary [M, MT, Tay2]. We adapt Melrose's proof of the Weyl law for manifolds with concave boundary to obtain the pointwise Weyl laws. In fact, they are simpler than the integrated Weyl laws of [M] and seem to be of independent interest. The simplest (non-compact) model cases are the exterior of the unit ball in \mathbb{R}^n and the Friedlander model (which is not a cosine wave kernel). The restriction $E_N(t, q, q)$ of the Neumann wave kernel of the exterior of the unit ball B_1 to the diagonal of $\partial B_1 \times \partial B_1$ is constant as q varies on the boundary, but involves special values of Hankel functions and is thus not elementary. Even this special case does not seem to have been studied before. The relevant background is given in Appendix B of [MT].

It appears plausible that Theorem 1.1 is valid for all Riemannian manifolds with smooth boundary, and possibly for manifolds with corners such as the Bunimovich stadium or a hyperbolic billiard. But such generalizations would require a different proof which does not make use of a parametrix

construction for the wave group of a manifold with boundary. In future work, we plan to use the scaling method Seeley and Melrose [M5] (in his proof of Ivrii's two term Weyl law [I]) in place of a parametrix construction to estimate sup norms of Cauchy data. However the diffractive case is more concrete and is of independent interest in illustrating the application of the Melrose-Taylor parametrix, and Melrose's technique for obtaining the remainder estimate in Weyl's law.

1.2. Surfaces of non-positive curvature with concave boundary. A special but important case of manifolds with concave boundary are those of non-positive curvature, obtained by removing some disjoint convex 'obstacles' from a non-positively curved manifold X . Thus,

$$M = X \setminus \bigcup_{j=1}^r \mathcal{O}_j, \quad (1.9)$$

where \mathcal{O}_j are embedded non-intersecting geodesically convex domains (or 'obstacles') \mathcal{O}_j . In the case where X is a flat torus or a square, such a billiard is often called a Lorentz-Sinai dispersing billiard. We refer to [CM] (see also ([JZ]) for background and references to the literature.

Billiards on (M, g) of the form (1.9) never have self-focal points. Self-focal points q are necessarily self-conjugate, i.e. there exists a broken Jacobi field along a geodesic billiard loop at q vanishing at both endpoints. But as shown in [JZ], non-positively curved dispersive billiards do not have conjugate points.

1.3. Example: complements of polar caps on spheres. A family of simple examples of Riemannian manifolds with concave boundary is given by certain complements of polar caps on standard spheres S^n . Let $S_r^n := S^n \setminus B_r(p)$ be the complement of a 'polar cap' of radius r . For $r < \pi/2$, S_r^n is a concave domain in S^n . When $r = \pi/2$, S_r^n becomes an upper hemisphere with totally geodesic (hence weakly concave) boundary. For the hemisphere, the Cauchy data of eigenfunctions saturate the remainder bounds and sup norm of Corollary 2. Indeed, the Neumann eigenfunctions of the upper hemisphere are restrictions of even eigenfunctions of S^n under the antipodal map $x \rightarrow -x$, while the Dirichlet eigenfunctions are restrictions of the odd ones. Even eigenfunctions are even degree spherical harmonics, and odd eigenfunctions are odd degree spherical harmonics. The extremal eigenfunctions are restrictions of zonal spherical harmonics with pole on the boundary; zonal means rotationally invariant for the axis through the pole (and its antipode). When $r < \pi/2$, so that the boundary is strictly concave, the sup norm bounds on boundary traces are not achieved. When $r > \pi/2$ the boundary is convex, and again the sup norm bounds on Cauchy data of eigenfunctions are not achieved. All types of extremal behavior (i.e. for all L^p norms) occur for the hemisphere.

We also remark that one can explicitly construct the Dirichlet or Neumann eigenfunctions on complements of polar caps using separation of variables. The radial factors are given by Legendre functions that may be singular at the poles. An example of an eigenfunction is the Green's function $G(\lambda, x, q)$ with pole q at the north pole of S_r^n with respect to a choice of maximal abelian subgroup of $SO(n+1)$. The Green's function satisfies $(\Delta + \lambda)G(\lambda, x, q) = 0$ in S_r^n since the δ_q lies outside the domain. It is rotationally invariant and therefore satisfies the Dirichlet boundary condition if λ is chosen so that $G(\lambda, r, q) = 0$ and satisfies Neumann boundary conditions if $\partial_r G(\lambda, r, q) = 0$. However, it is never an extremal eigenfunction.

1.4. A question. The example of polar caps suggests the question whether the general the bounds of Corollary 2 are only achieved when ∂M is totally geodesic. Of course, the example of polar caps provides only slim evidence. In particular, we ask if the sup norm bounds are ever achieved if the boundary is strictly concave. Note that concavity of the boundary implies non-existence of self-focal points on the boundary if the curvature is ≤ 0 .

2. WAVE FRONT SET BOUNDS ON CAUCHY DATA OF THE WAVE GROUP

The sup norm bounds of Proposition 1 and Theorem 1.1 are derived from an analysis of the singularities of the boundary trace of the wave kernel along the diagonal,

$$E_B^b(t, q, q) = \sum_{j=1}^{\infty} \cos(t\lambda_j) |\varphi_j^b(q)|^2. \quad (2.1)$$

In this section we calculate the wave front set of the boundary restriction (Cauchy data) of the wave group with Dirichlet or Neumann boundary conditions. The results were also stated and used in [JZ] and [Z4].

2.1. Wave front set of the wave group. We denote by

$$E_B(t) = \cos(t\sqrt{-\Delta_B}), \quad \text{resp.} \quad S_B(t) = \frac{\sin(t\sqrt{-\Delta_B})}{\sqrt{-\Delta_B}} \quad (2.2)$$

the even (resp. odd) wave operators (M, g) with boundary conditions B . The wave group $E_B(t)$ is the solution operator of the mixed problem

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta)E_B(t, x, y) = 0, \\ E_B(0, x, y) = \delta_x(y), \quad \frac{\partial}{\partial t} E_B(0, x, y) = 0, \quad x, y \in M; \\ BE_B(t, x, y) = 0, \quad x \in \partial M \end{cases}$$

The wave front sets of $E_B(t, x, y)$ and $S_B(t, x, y)$ are determined by the propagation of singularities theorem of [MSj] for the mixed Cauchy Dirichlet (or Cauchy Neumann) problem for the wave equation. We recall from

[Hör90] (Vol. III, Theorem 23.1.4 and Vol. IV, Proposition 29.3.2) that

$$WF(E_B(t, x, y)) \subset \bigcup_{\pm} \Lambda_{\pm}, \quad (2.3)$$

where $\Lambda_{\pm} = \{(t, \tau, x, \xi, y, \eta) : (x, \xi) = \Phi^t(y, \eta), \tau = \pm|\eta|_y\} \subset T^*(\mathbb{R} \times \Omega \times \Omega)$ is the graph of the generalized (broken) geodesic flow, i.e. the billiard flow Φ^t . The same is true for $WF(S_B)$. As mentioned above, the broken geodesics in the setting of (1.9) are simply the geodesics of the ambient negatively curved surface, with the equal angle reflections at the boundary; tangential rays simply continue without change at the impact.

2.2. Restriction of wave kernels to the boundary. The first tool in the proof of Theorem 1.1 is the analysis of the restriction of the Schwartz kernel $E_B(t, x, y)$ of $\cos t\sqrt{\Delta_B}$ to $\mathbb{R} \times \partial M \times \partial M$ and further to $\mathbb{R} \times \Delta_{\partial M \times \partial M}$, where $\Delta_{\partial M \times \partial M}$ is the diagonal of $\partial M \times \partial M$. We denote by dq the surface measure on the boundary ∂M , and by $ru = u|_{\partial M}$ the trace operator. We denote by $E_B^b(t, q', q) \in \mathcal{D}'(\mathbb{R} \times \partial M \times \partial M)$ the following boundary traces of the Schwartz kernel $E_B(t, x, y)$ defined in (2.2):

$$E_B^b(t, q', q) = \begin{cases} r_{q'} r_q \partial_{\nu_{q'}} \partial_{\nu_q} E_D(t, q', q), & \text{Dirichlet} \\ r_{q'} r_q E_N(t, q', q), & \text{Neumann} \end{cases} \quad (2.4)$$

The subscripts q', q refer to the variable involved in the differentiating or restricting.

2.3. Wave front set of the restricted wave kernel. The first and simplest piece of information is the wave front set of (2.1). It follows from (2.3) and from standard results on pullbacks of wave front sets under maps, the wave front set of $E_B^b(t, q, q')$ consists of co-directions of broken trajectories which begin and end on ∂M . That is,

$$\begin{aligned} WF(\gamma_q^B \gamma_{q'}^B E(t, q, q')) &\subset \{(t, \tau, q, \eta, q', \eta') \in B^* \partial M \times B^* \partial M : \\ &[\Phi^t(q, \xi(q, \eta))]^T = (q', \eta'), \tau = |\xi|\}. \end{aligned} \quad (2.5)$$

Here, the superscript T denotes the tangential projection to $B^* \partial M$. We refer to Section 2 of [HZ12] for an extensive discussion. It follows from (2.5) that

$$\begin{aligned} WF(\gamma_q^B \gamma_{q'}^B E(t, q, q)) &\subset \{(t, \tau, q, \eta, q, \eta') \in B_q^* \partial M \times B_q^* \partial M : \\ &[G^t(q, \xi(q, \eta))]^T = (q, \eta'), \tau = |\xi(q, \eta)|\}. \end{aligned} \quad (2.6)$$

Thus, for $t \neq 0$, the singularities of the boundary trace $\gamma_q^B \gamma_{q'}^B E(t, q, q)$ at $q \in \partial M$ to broken bicharacteristic loops based at q in \overline{M} . When $t = 0$ all inward pointing co-directions belong to the wave front set.

Remark 2.1. *One of the principal features of the boundary trace $E_B^b(t, q, q)$ along the diagonal is that the singularity at $t = 0$ becomes uniformly isolated from other singularities, while the interior kernel $E_B(t, x, x)$ has singularities at $t = 2d(x)$ arbitrarily close to $t = 0$.*

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3. PARAMETRICES FOR THE WAVE GROUP ON MANIFOLDS WITH CONCAVE BOUNDARY

To prove Proposition 1 and Theorem 1.1, we need more than wave front set bounds on the Cauchy data of Dirichlet or Neumann wave kernels $E_B(t, q, q')$ along the boundary. We will also need to have explicit control over the singularity of $E_B(t, q, q)$ at $t = 0$ and $q \in \partial M$, and approximate control over certain microlocalizations $a_h(q, D_q)E_B(t, q, q')|_{q=q'}$. To obtain such control, we use the Melrose-Taylor parametrix for $E_B(t, x, y)$ for wave kernels on manifolds with concave boundary [MT, M, Tay1, Tay2]. We only need the parametrix along $\partial M \times \partial M$, and this is a substantial simplification. We follow [M] closely in our analysis of the parametrix and the singularity at $t = 0$.

3.1. Kirchhoff formula. In this section we review the classical Kirchhoff formula for the fundamental solution of the wave equation. Background on Kirchhoff's formula in varying degrees of generality can be found in [Sob] (Lecture 14), Taylor's notes [Tay3] (section 2), Theorem 4.1.2 of [Fr], [F] (page 10). The Kirchhoff formulae show that the parametrix construction for the wave kernel can be reduced to that for the (Dirichlet-to-) Neumann operator \mathcal{N} and its inverse. This clarifies the relation between the parametrix constructions for Dirichlet vs. Neumann boundary conditions.

We assume as above that $(M, g) \subset (X, g)$ where (X, g) is a compact C^∞ Riemannian manifold without boundary. We denote the Schwartz kernels of $\exp it\sqrt{-\Delta_X}$ by $F(t, x, y)$ and of $\cos t\sqrt{\Delta_X}$ by $E_X(t, x, y)$. The Kirchhoff formula is a layer potential formula for the solution of a certain mixed Cauchy and boundary problem for the wave equation,

$$\begin{cases} \square u = 0, \\ u = g \text{ Dirichlet or } \partial_\nu u = h \text{ Neumann on } \Sigma \\ u = 0, \quad t \leq 0 \end{cases}.$$

The *Kirchhoff formula* is the Green's formula,

$$u(t, x) = \int_0^t \int_{\partial M} (\partial_{\nu_y} G(s, t, x, y)) u(s, y) - G(s, t, x - y) \partial_{\nu_y} u(s, y) dS(y) ds,$$

where G is the forward fundamental solution of \square on X ,

$$\begin{cases} G(t, s, x, y) = H(t - s) \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}, \\ G(-s, x, y) = 0, \quad s \geq 0. \end{cases}$$

Here, $H(t) = \mathbf{1}_{t \geq 0}$ is the Heaviside step function. The complication is that the formula involves both $u, \partial_\nu u$ on ∂M , but only half the data is prescribed by the equation. Given the data $u|_{\partial M \times \mathbb{R}}$,

$$\partial_{\nu_y} u(s, y) = \mathcal{N}(u|_{\partial M \times \mathbb{R}}), \quad \text{where } \mathcal{N} \text{ is the Neumann operator.}$$

We now apply the Kirchhoff formula to obtain expressions for the Dirichlet, resp. Neumann, wave kernels of (M, g) . They have the form

$$E_B(t, x, y) = F(t, x, y) + C_B(t, x, y)$$

where F is the free wave kernel of (X, g) and C_B is the compensating kernel, designed so that the sum satisfies the boundary condition. Thus,

$$\begin{cases} \square C_B(t, x, y) = 0, \\ C = -F \text{ Dirichlet or } \partial_\nu C = -\partial_\nu F \text{ Neumann on } \Sigma \\ C_B = 0, \quad t \leq 0 \end{cases}$$

LEMMA 4. *In the Neumann case,*

$$\begin{aligned} E_N(t, x, y) &= F(t, x, y) + \int_0^t \int_{\partial M} (\partial_{\nu_y} G(s, t, x, q')) \mathcal{N}^{-1} \partial_{\nu_y} F(s, q', y) \\ &\quad - G(s, t, x, q') \partial_{\nu_y} F(s, q', y) dS(q') ds. \end{aligned}$$

In the Dirichlet case,

$$\begin{aligned} E_D(t, x, y) &= F(t, x, y) + \int_0^t \int_{\partial M} (\partial_{\nu_y} G(s, t, x, q')) F(s, q', y) \\ &\quad - G(s, t, x, q') \mathcal{N} F(s, q', y) dS(q') ds. \end{aligned}$$

As the Lemma shows, the main difference between the Neumann and Dirichlet cases is that one needs to construct \mathcal{N}^{-1} in the Neumann case and \mathcal{N} in the Dirichlet case. See [Tay2] (section X.5), [F] or pages 262- 263 of [M] for more on this representation and on the Neumann operator and its inverse. Since there is a parametrix for $G(s, t, x, y)$ the main problem is to construct a parametrix for \mathcal{N} . We do not review more of the theory but head straight for Melrose's analysis of the parametrix in the case of concave boundary.

3.2. Microlocal decomposition. As on the bottom of page 261 of [M], we introduce coordinates $z = (x, y)$ near ∂M , where x is the defining function of ∂M and y is the normal variable. Then $\Delta = \frac{\partial^2}{\partial x^2} + Q(x, y, \partial_y)$, where the symbol $q(x, y, \eta)$ is the induced Riemannian norm on the level sets of x .

We recall (see [M, MT, Hör90, F]) that there exist two regions of $T^*\mathbb{R} \setminus \{0\} \times T^*\partial M$:

- the hyperbolic region(s) H_\pm where $\tau^2 > q(0, y, \eta)$,
- the elliptic region E where $\tau^2 < q(0, y, \eta)$,
- which are separated by the glancing hypersurfaces G_\pm where $\tau^2 = q(0, y, \eta)$, $\pm\tau > 0$.

As discussed on p. 265 of [M], a solution to the initial-boundary value problem

$$\begin{cases} \square w = 0, \\ \partial_\nu w|_{\partial M} = 0, \\ w(0, z) = w_0(z), \quad \partial_t w(0, z) = 0, \end{cases}$$

can be obtained by first solving the initial value problem and then adding compensating term so that the boundary condition is satisfied.

A solution to the initial value problem can be decomposed as $u = u_T^\pm + u_G$ (transversal plus glancing terms) where $u_G = u_G(t, x, y)$ has the form

$$u_G = \int e^{i\Phi(x, t, y, s, \tau, \eta) - i\Phi(x', 0, y', r, \tau, \eta)} c_G(x, t, y, s, r, \tau, \eta) d\tau d\eta dr ds, \quad (3.1)$$

where the phase has the form,

$$\Phi(x, t, y, s, \tau, \eta) = \alpha(x, t, y, \tau, \eta) + s|\eta|^{1/3}\beta(x, y, \tau, \eta) + \frac{1}{3}s^3|\eta|, \quad (3.2)$$

and where c_G is a first order classical symbol supported in $|s|, |r| \leq \varepsilon, |\tau^2 - |\eta|^2| \leq \varepsilon\tau^2$ for some $\varepsilon > 0$. The transversal term u_T^\pm is a standard Fourier integral operator.

It is shown in [M] (p. 266-267) that the Dirichlet or Neumann cosine wave kernel $w(t, q, q')$ can be constructed in this way as a sum of terms

$$w = w_1 + w_2^+ + w_2^-$$

with

$$\begin{cases} w_1 = (u_G)_c + w_{1,G} + w_{1,H}^+ + w_{1,H}^- + w_{1,E} \\ w_2^\pm = (u_H)_c + w_{2,G}^\pm + w_{2,H}^{\pm,+} + w_{2,H}^{\pm,-} + w_{2,E}^\pm. \end{cases} \quad (3.3)$$

Here u_c is the cutoff of the kernel $H(t)u(x, t, y; x', y')$ to $x < 0$ where x is the normal coordinate to ∂M . The hyperbolic and elliptic terms (with subscripts H, E) are of a standard kind (although the elliptic term is a Fourier integral operator with complex phase). In generalizing the arguments of [SZ02, SZ] to the boundary case, these terms do not require any essential modification and we therefore tend to suppress them in the exposition.

3.3. Glancing terms. The novel feature of the concave boundary case is the analysis of the glancing terms $w_{1,G}, w_{2,G}^\pm$. In (3.1) of [M], a parametrix for the glancing part is given in the form,

$$\begin{aligned} w_{1,G}(t, z, z'') &= \int \int_0^\infty e^{i\alpha(x,t,y,\tau',\eta') - it'(\tau' - \tau)\mu + is\beta'_0|\eta'|^{\frac{1}{3}} + i\frac{s^3}{3}|\eta'|} \\ &\quad e^{-i\alpha(x'',0,y'',\tau,\eta') - ir\beta''|\eta'|^{\frac{1}{3}} - i\frac{r^3}{3}|\eta'| \frac{aA(\beta) + bA'(\beta)}{A(\beta_0)}} \\ &\quad dt' ds dr d\tau d\tau' d\eta'. \end{aligned} \quad (3.4)$$

The s, r variable arise in (2.16) in the phase (3.2) defined near $\tau = \pm|\eta|$. Here, $x = 0$ defines ∂M and

- (1) α, β are real C^∞ functions, homogeneous of degree 1 resp. $\frac{2}{3}$ in (τ, η) . $\alpha(y, t, q, \tau, \eta)$ is linear in t , invariant under $(t, \tau) \rightarrow (-t, -\tau)$ and $\det(\partial_{t,y}\partial_{\tau,\eta}\alpha(0, t, y, \tau, \eta)) \neq 0$ on $\tau^2 = |\eta|^2$ ([M] (2.8), (2.10)). Hence

$$\begin{aligned} &-\alpha(0, t', y', \tau', \eta') + \alpha(0, t', y', \tau, \eta) \\ &= (y' + t'g)(\eta - \eta') + t'(\tau - \tau')\mu \end{aligned} \quad (3.5)$$

where g, μ are C^∞ and homogeneous of degree zero with $\mu\tau > 0$.

- (2) $\beta(y, q, \tau, \eta)$ is independent of t and along the boundary is given by $\beta(0, q, \tau, \eta) = \tau^2 - |\eta|^2|\eta|^{-\frac{4}{3}}$ ([M] (2.9) and above (3.2) ¹. β is homogeneous of degree $2/3$; note that the factor $|\eta|^{-\frac{4}{3}}$ is missing in (2.9) but corrected in (3.2). Then

$$\beta'_0 = (\tau^2 - |\eta'|^2)|\eta'|^{-\frac{4}{3}}, \quad \beta'' = \beta(x'', 0, y'', \tau, \eta'). \quad (3.6)$$

- (3) $\Phi(0, t, y, s, \tau, \eta) = \alpha(0, t, y, \tau, \eta) + s|\eta|^{\frac{1}{3}}\beta(0, y, \tau, \eta) + \frac{1}{3}s^3|\eta|$. [M] (2.16).

¹Note that $|\eta|^2$ is not the norm square with respect to the boundary metric, which is $q(0, y, \eta)$ above; as is stated at the top of p. 268 of [M], β_0 is independent of y .

- (4) a, b are not classical symbols but are supported near $s = r = 0, \tau^2 = |\eta|^2$ and have the form

$$\begin{aligned} \tilde{a}_G \sim & \sum_{j \geq 0} a_{j+1}(x, t, y, t', y', \tau, \eta) \Phi_{\pm}^{-1, j}(\beta(0, \tau, \eta)) \\ & + a_0(x, t, y, t', y', \tau, \eta) \end{aligned} \quad (3.7)$$

(with different coefficients for a resp. b) where a_j, b_j are classical symbols and $\Phi_{\pm}^{-1}(z) = \frac{A_{\pm}(z)}{A'_{\pm}(z)}$, and with $\Phi_{\pm}^{-1, j}(z) = \frac{d^j}{dz^j} \Phi(z)$. See (2.14) of [M].

- (5) u_G is given in (2.17) of [M] and u_c is obtained by cutting off u' in $x \leq 0$.
 (6) $A_{\pm}(s) = Ai(e^{\pm 2\pi i/3} s)$ where $Ai(s)$ is the Airy function that exponentially decreases as $s \rightarrow \infty$.

The corresponding term(s) for w_2^{\pm} have a similar form with Φ replaced by $\varphi_{\pm}(0, z', \zeta) = z' \cdot \zeta$ ([M], (2.1)). The cutoff term $(u_G)_c$ is also handled in the same way. Hence we focus on $w_{1,G}$.

4. POINTWISE WEYL LAWS ON THE BOUNDARY: PROOF OF PROPOSITIONS 1 AND 3

In this section we prove the pointwise Weyl law of Proposition 1. To prove Proposition 1, we show that the boundary trace $E_N^b(t, q, q)$ of the Neumann wave kernel has a normal singularity at $t = 0$. In the Dirichlet trace we take the normal derivative in each variable before restricting to the diagonal of the boundary. We follow [M] closely in his proof of the same assertion for the integral of $E_N(t, x, x)$ over the domain. In fact, the proof of the result for the restriction to the boundary is simpler. What we do is to go over the proof in [M] of the normality of the singularity at $t = 0$ of the integrated kernel and verify that the same steps prove normality of the un-integrated boundary trace.

The main step in the proof of Proposition 1 is to determine the singularity at $t = 0$ of

$$S_q(t) : = E_B^b(t, q, q) = \sum_j \cos t \lambda_j \left| \varphi_j^b(q) \right|^2. \quad (4.1)$$

PROPOSITION 5. *The diagonal boundary trace of the Neumann wave kernel, $E_N^b(t, q, q)$ or Dirichlet wave kernel $E_D^b(t, q, q)$ has a normal singularity at $t = 0$.*

To prove this and to determine the singularity coefficients, we study the dual problem

$$S_q(\lambda, \rho) = \int_{\mathbb{R}} \hat{\rho}(t) S_q(t) e^{it\lambda} dt \quad (4.2)$$

where $\rho \in \mathcal{S}(\mathbb{R})$ (Schwartz space) with $\hat{\rho} \geq 0$, $\hat{\rho} \in C_c^{\infty}(\mathbb{R})$ even, satisfying $\int_{\mathbb{R}} \rho dx = 1$ and with $\text{supp } \hat{\rho}$ contained in a sufficiently small neighborhood

$[-\varepsilon, \varepsilon]$ of $t = 0$ so that $t = 0$ is the only singularity of $E_B^b(t, q, q)$ in $\text{supp } \hat{\rho}$. To prove Proposition 5 and Proposition 1 it suffices to prove

PROPOSITION 6. *Let (M, g) be a compact Riemannian manifold with concave boundary. If $\text{supp } \hat{\rho}$ is sufficiently close to $t = 0$, then $S_q(\lambda, \rho)$ is a semi-classical Lagrangian distribution whose asymptotic expansion in both the Dirichlet and Neumann cases is given by*

$$S_q(\lambda, \rho) = \frac{\pi}{2} \sum_j (\rho(\lambda - \lambda_j) + \rho(\lambda + \lambda_j)) |\varphi_j^b(q)|^2 = C_n \lambda^n + C'_n A_n(q) \lambda^{n-1} + O(\lambda^{n-2}), \quad (4.3)$$

where $C_n = \frac{\omega_n}{(2\pi)^n}$, C'_n is a constant depending only on the dimension and $A_2(q)$ is the mean curvature of the boundary, i.e. the trace of the second fundamental form at q .

Here ω_n is the volume of the unit ball in \mathbb{R}^n . Proposition 5 implies Proposition 6 by a standard cosine Tauberian theorem (see the Appendix to [SV] or [Hör90]). We will not review this step but proceed to the proof of Proposition 5.

Before starting the proof we note that there are two independent aspects to the Proposition. The first is that $S_q(\lambda, \rho)$ has a complete asymptotic expansion when $\hat{\rho}$ has support sufficiently close to $t = 0$. The second is the calculation of the coefficients. The coefficients can be calculated by a heat kernel method; see Ozawa [O] for the calculation of the coefficients and for a one term expansion. A formal calculation of the first two coefficients is given in [BSS]. Assuming $S_q(\lambda, \rho)$ admits an expansion, the coefficients may be found by using the subordination formula and matching coefficients of $S_q(\lambda, \rho)$ with those of the heat kernel expansion.

4.1. The singularity at $t = 0$ of the restricted wave kernel. We follow the proof of the Weyl law for concave boundary in [M], but take boundary trace along the diagonal instead of integrating $E_N(t, x, x)$ over the domain. As mentioned above, there is a minimal ‘loop length’ ℓ of geodesic billiard loops in M which begin and end on ∂M . For $|t| < \ell$ the only singularity of $S_q(t)$ occurs at $t = 0$. We use the microlocal decomposition (3.3) into hyperbolic, elliptic and glancing sets. Away from the glancing (tangential) directions, $E_B^b(t, q, q)$ is a Fourier integral kernel whose symbol is computed in [HZ12]. To obtain the full remainder estimate of order $O(\lambda^{n-1})$ we need to determine the contribution of the glancing directions to the pointwise Weyl trace at $t = 0$. For the remainder of the proof we concentrate on the contribution of the glancing term.

To avoid excessive duplication of material in [M], we only prove that the $t = 0$ singularity of the glancing term $w_G(t, q, q)$ is normal. The proof is a relatively small modification of the proof of the Weyl law in [M], and we only explain the modification and not the entire proof.

Remark: Melrose assumes in (1.3) of [M] that for $t \neq 0$ the generalized broken geodesic flow Φ^t fixes no points of $S^*\partial M$, i.e. tangential directions to ∂M . He notes that it is a generic property of manifolds with concave boundary. We do not make the assumption here since the (M, g) which achieve maximal sup norm growth of eigenfunctions are also non-generic. Geodesic loops with base point on the boundary are defined by $\exp_x \xi = x$ and therefore the set of loops of length $\leq T$ (and therefore the loop directions with the loop length bound) form a closed set in $S^*_{\partial M} M$, the unit co-vectors with foot point on ∂M . If we assume $\text{Fix} \Phi^t|_{S^*\partial M} = \emptyset$ for $t \neq 0$, then there is a minimal positive angle between the initial and terminal directions of loops. Thus we could microlocalize away from the glancing set.

We first consider the Neumann case. We restrict $w_{1,G}$ to the boundary to obtain,

$$w_{1,G}(t, q, q'') = \int \int_0^\infty e^{i\alpha(0,t,q,\tau',\eta') - it'(\tau' - \tau)\mu + is\beta'_0|\eta'|^{\frac{1}{3}} + i\frac{s^3}{3}|\eta'|} e^{-i\alpha(0,0,q'',\tau,\eta') - ir\beta''|\eta'|^{\frac{1}{3}} - i\frac{r^3}{3}|\eta'|} \frac{aA(\beta) + bA'(\beta_0)}{A(\beta_0)} \quad (4.4)$$

$$dt' ds dr d\tau d\tau' d\eta'.$$

The normality of the singularity at $t = 0$ is standard for the elliptic and hyperbolic terms, so we only discuss the glancing terms $(u_G)_c, w_{1,G}, w_{2,G}^\pm$. The key point is the following

LEMMA 7.

$$w_{1,G}(t, z, z) = \int e^{it\theta} e(t, \theta, z) d\theta,$$

with $e(t, \theta, z)$ a classical symbol when $z = q \in \partial M$. The amplitudes a, b may be taken to be supported in $|s|, |r| \leq \varepsilon, |\tau^2 - |\eta|^2| \leq \varepsilon\tau^2$ and the contribution of the glancing term to the first two terms of (6) are of order $O(\varepsilon)$ as $\varepsilon \rightarrow 0$.

Further,

$$w_{1,G}^{(2)}(t, q, q) = \int e^{i\alpha(0,t,q,\tau',\eta') - i\alpha(0,0,q,\tau',\eta')} As(\beta, \beta_0) dr d\tau' d\eta'.$$

To prove Lemma 7, we start from (3.5) of [M] but do not integrate in z . We first analyze the harder term $w_{1,G}$. We use the expressions (3.2)-(3.3)-(3.4) of [M] for $w_{1,G}$. Here the $dt' d\tau$ integrations have already been carried out. In the case of the Neumann kernel, we simply restrict (3.3) of [M] to the boundary to get

$$w_{1,G}^{(1)}(t, q, q) = \int_{\tau'(s-r) \leq 0} e^{i\alpha(0,t,q,\tau',\eta') - i\alpha(0,0,q,\tau',\eta') + is\beta_0|\eta'|^{\frac{1}{3}} + is^3\frac{|\eta'|}{3} + ir\bar{\beta} - ir^3|\eta'|/3} As(\beta, \beta_0) ds dr d\tau' d\eta'. \quad (4.5)$$

Here,

$$As(\beta, \beta_0) = (aA(\beta) + bA'(\beta))/A(\beta_0),$$

where $A(\beta)$ is a special value of the Airy function. Also, on the boundary $x'' = 0$, $\beta = \beta_0$ ([M], p. 268). Hence the restriction of the Airy part of the amplitude to the boundary has the form,

$$As(\beta, \beta_0)|_{q=q' \in \partial M} = a + b \frac{A'}{A}(\beta_0). \quad (4.6)$$

Here, a, b are not classical symbols but have the form (2.14) of [M]. Also, α is a certain phase function whose precise nature we will not need to know.

The next step in [M] is to change variable to $\theta = \partial_t \alpha$, $\lambda = \beta_0 |\theta|^{-2/3}$, $S = s |\eta'|^{-1/3}$, $R = r |\eta'|^{-1/3} |\theta|^{1/3}$. The expression for the boundary trace is then the analogue of (3.6) of [M],

$$e(t, q, \theta) = \int \int_{\theta(S-R) \leq 0} \int_{\mathbb{R}} e^{i(S\lambda + S^3/3 - R\lambda - R^3/3)|\theta|} As(\lambda|\theta|^{2/3}, \lambda|\theta|^{2/3}) dR dS d\lambda. \quad (4.7)$$

This is the essential starting point for the proof of Lemma 7. The goal is to show that $e(t, q, \theta)$ is a classical symbol in θ . Here, $\theta \in \mathbb{R}$ and there are two sides accordingly as $\theta > 0, \theta < 0$. The amplitudes a, b are supported in a region where both S, R are close to zero. See (3.9) in which the amplitude is denoted by a . The sign of λ is that of β_0 .

The Airy amplitude is a symbol of the form,

$$As(\lambda|\theta|^{2/3}, \lambda|\theta|^{2/3}) \sim \sum_j c_j(\lambda, 0) \theta^{\frac{1}{3}-j}, \quad |\lambda\theta^{2/3}| \geq 1.$$

We change variables to $U = S - R, V = S + R$ to get

$$e(t, q, \theta) \sim \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty e^{i(\pm U\lambda + UQ_2(U, V))|\theta|} a(U, V, \lambda, \theta) d\lambda dU dV, \quad (4.8)$$

where $a(U, V, \lambda, \theta)$ is a symbol in θ with compact support in (λ, U, V) near $(0, 0, 0)$. Here, U runs over $[0, \infty)$ and λ, V run over \mathbb{R} .

The phase is

$$\Psi(\lambda, U, V) = \lambda U + UQ_2(U, V),$$

where

$$Q_2(U, V) = 3U^2 + V^2.$$

The critical point equations in (λ, U, V) are

$$\partial_\lambda \Psi = U = 0, \quad \partial_U \Psi = \lambda + V^2 + 9U^2 \geq 0, \quad \partial_V \Psi = 2UV.$$

The only critical point is $(\lambda, U, V) = (0, 0, 0)$ but the phase is degenerate. It is non-degenerate in (λ, U) , but $U \geq 0$ in the domain of integration.

Remark:

There is a general stationary phase theorem in Hörmander vol 1 (Theorem 7.7.17) in a half-space $x_n < 0$. In our case, $x_n = S - R = U$. The x' variable

is therefore λ and unfortunately the phase is linear in λ so that result does not apply.

Melrose takes advantage of the fact that $U > 0$ by making an almost analytic extension of the $d\lambda$ integral to the upper half space $\lambda + i\sigma, \sigma \geq 0$. Applying Stokes' theorem, we obtain

$$e(t, x, \tau) = \int_{\mathbb{C}_+} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\theta(U(\lambda+i\sigma)+i/3UQ_2(U,V))} \bar{\partial}_{\lambda+i\sigma} \tilde{A}(\lambda+i\sigma)|\theta|^{\frac{2}{3}} d\lambda d\sigma dU dV, \quad (4.9)$$

where \tilde{A} is an Airy symbol of the form,

$$a \frac{A_{\pm}(\beta_0)}{A'_{\pm}(\beta_0)} + b \frac{A'_{\pm}(\beta_0)}{A'_{\pm}(\beta_0)}.$$

A key point of the almost analytic extension is that $\bar{\partial}_{\lambda+i\sigma} \tilde{A}(\lambda+i\sigma)|\theta|^{\frac{2}{3}}$ vanishes to infinite order at $\sigma = 0$. Moreover, the Airy quotients have uniform asymptotic expansions in the half plane when multiplied by functions vanishing to all orders at $\sigma = 0$. Thus the Airy symbol is classical in the half-plane. As in (3.9) of [M], one concludes that

$$e(t, q, \theta) = \int_{\sigma\theta \leq 0} \int_{\theta U \leq 0} e^{i|\theta|(U(\lambda+i\sigma)+\frac{1}{3}UQ_2(U,V))} a d\sigma d\lambda dU dV, \quad (4.10)$$

where a is a classical symbol in θ which is compactly supported near $U = V = \lambda + i\sigma = 0$ and vanishes to infinite order at $\sigma = 0$. We now use (4.10) as a new starting point to prove Lemma 7. We restate the result as,

LEMMA 8. *$e(t, q, \theta)$ is a classical symbol in θ in the sense that*

$$e(t, q, \theta) \sim \sum_{j=0}^M e_j(t, q) \theta^{-j} + R_M(t, q, \theta),$$

where

$$D_{\theta}^m R_M(t, q, \theta) \leq C(t, x) \theta^{-M-m}.$$

To prove symbol estimates we expand

$$a = \bar{\partial}_{\lambda+i\sigma} \tilde{A}(\lambda+i\sigma)|\theta|^{\frac{2}{3}} \simeq \sum_{j=0}^M \tilde{a}_j(\lambda+i\sigma, U, V) |\theta|^{-j} + R_M, \quad (4.11)$$

which can be differentiated any number of times, with \tilde{a}_k vanishing to infinite order at $\sigma = 0$.

We then integrate by parts in the dU integral using the fact that

$$\frac{\partial}{\partial U} (U(\lambda + i\sigma + Q_2(U, V))) = \lambda + i\sigma + d_U(UQ_2(U, V)).$$

We form the integration by parts operator

$$L_U = \theta^{-1} \frac{1}{\lambda + i\sigma + d_U(UQ_2(U, V))} \frac{\partial}{\partial U},$$

which we may use away from the critical point $U = V = \lambda + i\sigma = 0$. In fact, the operator is well defined as long as $\sigma \neq 0$. Moreover, since the amplitude vanishes to infinite order at $\sigma = 0$, we can integrate by parts any number of times on the full domain $\sigma \geq 0$. Each partial integration picks up a boundary term at $U = 0$. At $U = 0$, the integral ceases to be oscillatory since the whole phase has a factor of U and vanishes when $U = 0$. The classical expansion of the amplitude (4.11) then gives the symbol expansion of $e(t, q, \theta)$. Each partial integration introduces a factor of θ^{-1} and therefore the remainder is formed by the remainder of the symbol expansion of the amplitude and by the remainder in an arbitrary number M of partial integrations in L_U . This leaves an arbitrarily high power of θ^{-1} times an integral of the same form. The limit as $\theta \rightarrow \infty$ of the integral equals zero by dominated convergence, since the integrand tends to zero as long as $U > 0$. This completes the proof of Lemma 8 and therefore Lemma 7.

4.2. The remaining terms. To complete the proof of the Proposition we must also deal with a second term $w_{1,G}^{(2)}(t, q, q)$ of $w_{1,G}$ (see (3.4) of [M]) as well as the terms $w_{2,G}^\pm$ ([M] (2.26) and $(u_G)_c$ (2.21).

In the second term $w_{1,G}^{(2)}(t, q, q)$ of $w_{1,G}$ there is no $drds$ integral. Integrating out dr and restricting to the boundary produces

$$w_{1,G}^{(2)}(t, q, q) = \int e^{i\alpha(0,t,q,\tau',\eta') - i\alpha(0,q,\tau',\eta')} As(\beta_0, \beta_0) d\tau' d\eta'. \quad (4.12)$$

This gives an integral of the form,

$$\int_{\mathbb{R}} As(\lambda|\theta|^{2/3}, \lambda|\theta|^{2/3}) a(\lambda) d\lambda$$

and we obtain a classical expansion using the classical expansion of $As(\lambda|\theta|^{2/3}, \lambda|\theta|^{2/3})$.

The terms $w_{2,G}^\pm$ have the same form as $w_{1,G}$ and the normality of the singularity at $t = 0$ is proved in the same way.

The term u_c is obtained from u_G (3.1) ([M] (2.17)) by multiplying by $H(t)$ and cutting off in $x < 0$. As explained on page 270 of [M], the integral of $(u_G)_c$ over $z = z'$ is classical. We now verify that the boundary trace u_G^b on the diagonal of the boundary is classical. It is given by

$$u_G^b(t, q, q) = \int e^{i\Phi(0,t,q,s,\tau,\eta) - i\Phi(0,0,q,r,\tau,\eta)} c_G(0, t, q, s, r, \tau, \eta) d\tau d\eta dr ds. \quad (4.13)$$

As above, $\Phi(0, t, y, s, \tau, \eta) = \alpha(0, t, y, \tau, \eta) + s|\eta|^{\frac{1}{3}}\beta(0, y, \tau, \eta) + \frac{1}{3}s^3|\eta|$ and c_G is a first order classical symbol supported in $|s|, |r| \leq \varepsilon, |\tau^2 - |\eta|^2| \leq \varepsilon\tau^2$ for some $\varepsilon > 0$. This form is similar to that of $w_{1,G}$ but does not have the Airy amplitude. The proof of the normality of the singularity at $t = 0$ of $w_{1,G}$ applies to it as well.

4.3. Estimate of the glancing contribution: Proof of (1.7).

Proof. If the symbol is supported in the complement of an ε -conic neighborhood of the glancing set, then we may use only the hyperbolic and elliptic parts of the parametrix of the wave kernel in (3.3) and calculate the point-wise asymptotics as in [SZ02, SZ].

On the other hand if the symbol is supported in an ε -conic neighborhood of the glancing set, we use Lemma 8 and the fact that the amplitudes a, b are supported in a region where both $|s|, |r| \leq \varepsilon, |\tau^2 - |\eta|^2| \leq \varepsilon\tau^2$. It follows that the terms obtained as boundary values in the integration by parts are also of order ε . □

4.4. Proof of Proposition 3. The proof is very similar to that of Proposition 1 (and Proposition 5). Proposition 3 is again proved using a cosine Tauberian theorem, and the main step is to show that

$$S_a(t, q, q) := \sum_j |Op_h(a)\varphi_j^b(y)|^2 \cos t\lambda_j \quad (4.14)$$

has a normal singularity at $t = 0$. We have,

$$S_a(t, q, q) = Op_h(a)E^b(t, \cdot, \cdot)Op_h(a)^*|_{q=q'} \quad (4.15)$$

or with normal derivatives in the Dirichlet case.

Thus the only modification to the proof of Proposition 1 is that we apply $Op_h(a)$ under the integral sign in both variables for the parametrices for the wave kernel and their boundary traces. To prove the first statement of the Proposition, when the symbol vanishes near the glancing direction, there is a standard Fourier integral parametrix and one may apply the pseudo-differential operator $Op_h(a)$ to the oscillatory integral. By the “fundamental asymptotic expansion Lemma” of as [Tay1] (Section VIII §7), application of $Op_h(a)$ changes the amplitude to another amplitude with the same symbolic properties. Thus, in the elliptic or hyperbolic regions, where the oscillatory integral satisfies the assumptions (2.3)-(2.4) of Taylor (loc.cit.), the the oscillatory integrals (4.15) are standard ones.

For the glancing term we need to apply $Op_h(a)$ on the left and right to $w_{1,G}(t, q, q')$ (4.4) for $q = (0, y), q'' = (0, y'') \in \partial M$, and then set $q = q'$. Thus, we are applying semi-classical pseudo-differential operators on a manifold without boundary to an oscillatory integral of the form (4.16) with $x = x'' = 0$, and taking into (2) $\beta(0, q, \tau, \eta) = (\tau^2 - |\eta|^2)|\eta|^{-4/3}$ into account,

$$\begin{aligned} w_{1,G}(t, (0, y), (0, y'')) &= \int \int_0^\infty e^{i(\alpha(0, t, y, \tau', \eta') - \alpha(0, 0, y'', \tau, \eta')) - it'(\tau' - \tau)\mu} \\ &\quad e^{is\beta'_0|\eta'|^{\frac{1}{3}} + i\frac{s^3}{3}|\eta'|} e^{-ir\beta''|\eta'|^{\frac{1}{3}} - i\frac{r^3}{3}|\eta'|} \frac{aA(\beta) + bA'(\beta)}{A(\beta_0)} \\ &\quad dt' ds dr d\tau d\tau' d\eta'. \end{aligned} \quad (4.16)$$

where $\beta'_0 = \beta(0, q, \tau, \eta')$ and $\beta'' = \beta(0, q'', \tau, \eta')$ are defined in (3.6). They are independent of (q, q') . The terms of the phase $\alpha(0, t, y, \tau', \eta') - \alpha(0, 0, y'', \tau, \eta')$ and $t'(\tau - \tau)\mu$ are classical. Hence $Op_h(a)$ in either q or q' is the application of a pseudodifferential operator with a Fourier-Airy integral operator with a classical phase but with an Airy amplitude, which is a non-classical symbol. The fundamental asymptotic expansion lemma is essentially the stationary phase method, and it applies to this composition in the (q, q') variables. Thus, after application of $Op_h(a)$ on either side we obtain a new Fourier Airy integral operator on ∂M with the same phase but a new amplitude, with the same properties as those of $w_{1,G}$ and of the same order. The composition of Airy operators and the symbol expansion is discussed in detail in [M3, M4]. The rest of the argument proceeds as in the proof of Proposition 1.

Remark: Boundary operators induced by Fourier-Airy integral operators are discussed in Section 7 of [M3].

4.5. Dirichlet boundary conditions. To complete the proof of Proposition 5 we need to discuss the modifications in the case of Dirichlet boundary conditions. In the notation of [M], we now take $\partial_x \partial_{x''}$ and set $x = x'' = 0$. This changes the amplitude in several ways. From the phase, differentiation brings down a factor of

$$(I) \quad \partial_x \alpha(x, t, y, \tau', \eta'), \quad \partial_{x''} \alpha(x'', 0, y, \tau, \eta').$$

If the derivative is placed instead on the amplitude, we first have the term

$$(II) \quad \partial_{x''} \beta(x'', 0, y'', \tau, \eta') (-ir|\eta'|^{1/3}).$$

Finally we also have

$$(III) \quad \partial_x \partial_{x''} \frac{aA(\beta) + bA'(\beta)}{A(\beta_0)}.$$

After the differentiation we set $x = x'' = 0$.

Thus, in the Dirichlet case, we have

$$\begin{aligned} \partial_x \partial_{x''} w_{1,G}(t, q, q'') &= \int \int_0^\infty e^{i\alpha(0, t, q, \tau', \eta') - it'(\tau' - \tau)\mu + is\beta'_0|\eta'|^{1/3} + i\frac{s^3}{3}|\eta'|} \\ &\quad e^{-i\alpha(0, 0, q'', \tau, \eta') - ir\beta''|\eta'|^{1/3} - i\frac{r^3}{3}|\eta'|} \\ &\quad (\partial_x \alpha(x, t, y, \tau', \eta') + \partial_{x''} \alpha(x'', 0, y, \tau, \eta')) \\ &\quad + \partial_{x''} \beta(x'', 0, y'', \tau, \eta') (-ir|\eta'|^{1/3}) \frac{aA(\beta) + bA'(\beta)}{A(\beta_0)} \\ &\quad + \partial_x \partial_{x''} \frac{aA(\beta) + bA'(\beta)}{A(\beta_0)} \\ &\quad dt' ds dr d\tau d\tau' d\eta'. \end{aligned}$$

The new amplitude has the same essential properties as that of the Neumann case, since the only difficulty is with the Airy terms. However, the term (III) is still classical when restricted to $q = q' \in \partial M$. Thus, as explained in [M], the argument of the Neumann case extends to the Dirichlet case with no essential change.

4.6. Appendix on Airy functions. Here we recall the basic definitions and facts regarding Airy functions, referring to [MT] for background.

$$Ai(z) = \frac{1}{2\pi i} \int_L e^{v^3/3 - zv} dv,$$

where L is any contour that begins at a point at infinity in the sector $-\pi/2 \leq \arg(v) \leq -\pi/6$ and ends at infinity in the sector $\pi/6 \leq \arg(v) \leq \pi/2$. In the region $|\arg z| \leq (1 - \delta)\pi$ in $\mathbb{C} - \{\mathbb{R}_-\}$ write $v = z^{1/2} + it^{1/2}$ on the upper half of L and $v = z^{1/2} - it^{1/2}$ in the lower half. Then

$$Ai(z) = \Psi(z) e^{-\frac{2}{3}z^{3/2}}$$

with

$$\Psi(z) \sim z^{-1/4} \sum_{j=0}^{\infty} a_j z^{-3j/2}, \quad a_0 = \frac{1}{4}\pi^{-3/2}.$$

Set

$$A_{\pm}(s) = Ai(e^{\pm \frac{2\pi i}{3}} s).$$

Let

$$\Phi_{\pm}(z) = \frac{A'_{\pm}(z)}{A_{\pm}(z)}.$$

One has

$$A'_{\pm}(z) Ai(z) - Ai'(z) A_{\pm}(z) = c_{\pm}$$

hence

$$\Phi_{\pm} - \Phi i(z) = c_{\pm} [A_{\pm}(z) Ai(z)]^{-1}.$$

The Airy quotients satisfy the nonlinear ODE

$$\Phi'(z) = z - \Phi(z)^2, \quad \Phi = \Phi i(z), \text{ or } \Phi_{\pm}(z).$$

Also

$$\Phi_{\pm}(z) = \omega^{\mp 2} \Phi i(\omega^{\mp 2} z).$$

The poles of Φ_{\pm} lie on $e^{-i\pi/3}[-s_0, \infty]$ in the fourth quadrant. The poles of $\Phi_{-}(z)$ lie on $e^{i\pi/3}[-s_0, \infty]$ in the first quadrant. Outside any conic neighborhood of these rays,

$$\Phi_{\pm}(z) \sim z^{1/2} \sum_{j=0}^{\infty} b_j^{\pm} z^{-3j/2}, \quad |z| \rightarrow \infty.$$

One has

$$\Phi_{+}(z) = \overline{\Phi_{-}(\bar{z})}, \quad b_0^{\pm} = 1.$$

The standard Airy quotient Φ is a classical symbol of order $\frac{1}{2}$.

$$|D^j \frac{A'}{A}(\zeta)| \leq C_j (1 + |\zeta|)^{\frac{1}{2}-j}.$$

In fact,

$$\frac{A'}{A}(\zeta) \sim \sum_{j \geq 0} a_j^\pm \zeta^{\frac{1}{2} - \frac{3j}{2}}, \quad \text{Re } \zeta \rightarrow \infty.$$

5. SUP NORM BOUNDS FOR CAUCHY DATA: PROOF OF THEOREM 1.1

We now complete the proof of Theorem 1.1, following the outline of that in [SZ02]. We consider the remainder terms of the pointwise Weyl law of Proposition 1 and Corollary 2. The key point is the

PROPOSITION 9. *If the set of billiard loops at $q \in \partial M$ has measure zero in $B_q^* \partial M$, then given $\varepsilon > 0$, we can find a ball B centered at q and a $\Lambda < \infty$ so that for $\lambda \geq \Lambda$,*

$$\begin{cases} |R_D(\lambda, q)| \leq \varepsilon \lambda^{n-1}, & q \in B, \\ |R_N(\lambda, q)| \leq \varepsilon \lambda^{n-1}, & q \in B. \end{cases} \quad (5.1)$$

Before proving the Proposition, we show that it implies Theorem 1.1.

Indeed, we observe that for fixed $q \in \partial M$ and any $\varepsilon > 0$ then one can find a neighborhood $\mathcal{N}_\varepsilon(q)$ of q and an $\Lambda_\varepsilon(q)$ so that when $\lambda \geq \Lambda_\varepsilon(q)$ and $y \in \mathcal{N}_\varepsilon(q)$ we have $|R(\lambda, y)| \leq \varepsilon \lambda^{n-1}$. This implies that $|\varphi_j^b(y)| \leq \varepsilon \lambda^{(n-1)/2}$ if $y \in \mathcal{N}_\varepsilon(q)$ and $\lambda \geq \Lambda_\varepsilon(q)$. Since M is compact and since the $\mathcal{N}_\varepsilon(q)$ form open cover of M , we may choose a finite subcover and extract the largest $\Lambda_\varepsilon(q)$. For this Λ_ε , Corollary 2 gives

$$|\varphi_j^b(q)| \leq \varepsilon \lambda^{(n-1)/2}, \quad \lambda \geq \Lambda_\varepsilon.$$

Since Λ_ε depends only on ε , it follows that $\sup_{q \in \partial M} |\varphi_j^b(q)| = o(\lambda^{(n-1)/2})$, as stated in Theorem 1.1.

We now prove Proposition 9.

We introduce a cutoff $\hat{\rho} \in C_0^\infty(\mathbb{R})$, which as above is a positive even function such that $\hat{\rho}$ is identically 1 near 0, has support in $[-1, 1]$ and is decreasing on \mathbb{R}_+ . We also define ρ_T by $\hat{\rho}_T(t) = \hat{\rho}(\frac{t}{T})$, so that $\text{supp } \hat{\rho}_T \subset (-T, T)$. To prove (5.1) it suffices to prove

LEMMA 10. *If $|\mathcal{L}_q| = 0$, then for all $\varepsilon > 0$ there exists a neighborhood $\mathcal{N}(q, \varepsilon)$ and a time $T_0(\varepsilon)$ so that for $T \geq T_0(\varepsilon)$,*

$$\sum_{j=0}^{\infty} \rho(T(\lambda - \lambda_j)) (\varphi_j^b(q'))^2 \leq \varepsilon \lambda^{n-1}, \quad \text{if } q' \in \mathcal{N}(q, \varepsilon), \text{ and } \lambda \geq \Lambda. \quad (5.2)$$

Proof. We consider the smoothed restricted Weyl sum (4.3)

$$\begin{aligned} S_q(\lambda, \rho_T) &:= \rho_T * d_\lambda \Pi_{[0, \lambda]}(q, q) = \int_{\mathbb{R}} E_B^b(t, q, q) \hat{\rho}\left(\frac{t}{T}\right) e^{-it\lambda} dt \\ &= \sum_j (\rho(T(\lambda - \lambda_j)) + \rho(T(\lambda + \lambda_j))) |\varphi_j^b(q)|^2. \end{aligned} \quad (5.3)$$

As in [SZ02], the $\rho(T(\lambda + \lambda_j))$ term contributes $\mathcal{O}(\lambda^{-M})$ for all $M > 0$ and therefore may be neglected. To prove Lemma 10 it suffices to show that for any T ,

$$|S_{q'}(\lambda, \rho_T)| \leq \varepsilon \lambda^{n-1} + O_T(\lambda^{n-2}), \quad q' \in \mathcal{N}(q, \varepsilon). \quad (5.4)$$

If $|\mathcal{L}_q| = 0$, then

$$\mathcal{L}_q^T = \{\xi \in S_{q, in}^* M : \Phi_t(q, \xi) \in S_q^* M \text{ for some } t \in [-T, T] \setminus \{0\}\}$$

is closed and of measure zero. For a given $T > 0$, we can therefore construct a pseudodifferential cutoff

$$\chi_T(q, D) : L^2(\partial M) \rightarrow L^2(\partial M)$$

with the property that χ_T is microsupported in the set where $L^*(q, \eta) >> T$. Thus, given $\varepsilon_0 > 0$, we can find a $\chi_T \in C^\infty(B^* \partial M)$ so that

$$0 \leq \chi_T \leq 1, \quad \int_{B^* \partial M} (1 - \chi_T) d\sigma < \varepsilon_0, \quad \mathcal{L}_x^T \cap \text{supp } \chi_T = \emptyset. \quad (5.5)$$

In fact, we may construct $\chi_T(q, \eta) \in C_0^\infty(B^* \partial M)$ so that

$$\int_{B^* \partial M} (1 - \chi_T)(q, \eta) d\sigma(\eta) \leq 1/T^2, \quad (5.6)$$

and

$$|L^*(q, \eta)| \leq 1/T, \quad \text{on } \mathcal{N} \times \text{supp } B,$$

where \mathcal{N} is a neighborhood of q . We also denote by $\chi(q, D)$ its quantization as a semi-classical pseudo-differential operator on ∂M with Planck constant \hbar defined by $\hbar = \lambda_j^{-\frac{1}{2}}$.

We then study $\Pi_{[0, \lambda]}^b(q, q)$ by writing it in the form:

$$\begin{aligned} \Pi_{[0, \lambda]}^b(q, q) &= [(\chi_T(q, D) + (I - \chi_T(q, D))) \circ \Pi_{[0, \lambda]}^b \circ \\ &\quad \circ (\chi_T(q, D) + (I - \chi_T(q, D)))](q, q). \end{aligned} \quad (5.7)$$

There are two types (I, II) of terms among the four in (5.7). The first type I (of which there are three terms) has at least one factor of χ_T (on either side of $\Pi_{[0, \lambda]}^b$). The second type (of which there is just one term) has the form

$$(I - \chi_T(q, D)) \circ \Pi_{[0, \lambda]}^b \circ (I - \chi_T(q, D))(q, q). \quad (5.8)$$

The first type of term can be dealt with entirely by wave front set considerations. The second is more complicated but can be dealt with by Lemma 7.

We break up the analysis into one for short times and one for the rest. We fix an even function

$$\beta \in C_0^\infty(\mathbb{R}) \text{ which equals one on } [-2\delta, 2\delta].$$

The analysis for small times $[-\delta, \delta]$ involves the short time parametrix and pointwise Weyl laws discussed in §4 and §4.4 while for times $|t| \geq \delta$ it is not necessary to construct a parametrix.

5.1. Terms of both type I and II for $|t| \leq \delta$. Here we do not gain from using the cutoff χ_T since δ is less than the minimal loop length and the only singularity is at $t = 0$. We must use the analysis of the singularity at $t = 0$ in §4 and §4.4.

SUBLEMMA 11. *For $T \geq 1$,*

$$\left| \frac{1}{2\pi T} \int \beta(t) \rho(t/T) E_B^b(t, y, y) e^{-it\lambda} dt \right| \leq CT^{-1} \lambda^{n-1}.$$

Proof. For $T \geq 1$,

$$\left| \frac{1}{2\pi T} \int \beta(t) \rho(t/T) e^{it\tau} dt \right| \leq C_N T^{-1} (1 + |\tau|)^{-N}, \quad N = 1, 2, 3, \dots$$

Hence,

$$\begin{aligned} \left| \frac{1}{2\pi T} \int \beta(t) \rho(t/T) E_B^b(t, q, q) e^{-it\lambda} dt \right| \\ \leq CT^{-1} \sum_{j=0}^{\infty} (1 + |\lambda - \lambda_j|)^{-n-1} (\varphi_j^b(q))^2 \end{aligned}$$

We note that

$$\begin{aligned} \sum_{j=0}^{\infty} (1 + |\lambda - \lambda_j|)^{-n-1} (\varphi_j^b(q))^2 &= \int (1 + (\lambda - \mu)^{-n-1} d_\mu \Pi_{[0, \mu]}^b(q, q)) \\ &= \int (1 + (\lambda - \mu)^{-n-1} d_\mu (\mu^n + R(\mu, q))) \\ &= n \int (1 + (\lambda - \mu)^{-n-1} \mu^{n-1} + \\ &\quad + (n-1) \int (1 + (\lambda - \mu)^{-n-2} |R(\mu, q)| d\mu) \\ &\leq 2n \int (1 + (\lambda - \mu)^{-n-2} \mu^{n-1} d\mu \end{aligned}$$

In the second to last line we use the remainder estimate in the pointwise Weyl law in Proposition 1. \square

5.2. Terms of type I for $|t| \geq \delta$. In this section we prove,

SUBLEMMA 12.

$$\frac{1}{2\pi T} \int (1 - \beta(t)) \rho(t/T) (\chi_T(q, D) \circ E_B^b)(t, y, y) e^{-it\lambda} dt = O_{B,T}(1). \quad (5.9)$$

Proof. The estimate follows immediately from the fact that

$$E^b(t)\chi_T(q, D)^*(q, q), \quad \chi_T(q, D) \circ E^b(t, q, q) \in C^\infty(0, T). \quad (5.10)$$

By assumption,

$$\min_{\eta \in \text{supp } \chi} d_{\partial M}(q, \beta^k(q, \eta)) > 0, \quad \sum_{j=0}^k T^{(j)}(q, \eta) < T$$

and so there must be a neighborhood \mathcal{N} of q in ∂M so that if $q' \in \mathcal{N}$ then

$$\{\beta^j(q', \eta), 0 < j \leq k \text{ s.t. } \sum_{j=0}^k T^{(j)}(q, \eta) < T\} \notin \mathcal{N}, \quad \text{if } \eta \in \text{supp } \chi.$$

Therefore by (2.5)-(2.6),

$$(\chi_T(q, D) \circ E_B^b)(t, y, y) \in C^\infty(\{\delta \leq |t| \leq T\} \times M). \quad (5.11)$$

Taking the Fourier transform of the smooth function (5.10) completes the proof. \square

5.3. Terms of type II for $\delta \leq |t| \leq T$.

SUBLEMMA 13.

$$\left| \frac{1}{2\pi T} \int (1 - \beta(t)) \rho(t/T) E_B^b(t, y, y) e^{-it\lambda} dt \right| \leq C_T \sqrt{\varepsilon_0} \lambda^{n-1} + O_{b,T}(\lambda^{n-2}). \quad (5.12)$$

Proof. In view of SubLemma 12 we may insert $(I - \chi_T(q, D))$ to the left of $E_B^b(t)$.

We define

$$m_{T,\beta}(\tau) = \frac{1}{2\pi T} \int (1 - \beta(t)) \rho(t/T) e^{it\tau} dt.$$

The left side of Sublemma 13 equals

$$\left| \sum_{j=0}^{\infty} m_{T,\beta}(\lambda - \lambda_j) (I - \chi_T(q, D)) \varphi_j^b(y) \varphi_j^b(y) \right|.$$

Since $m_{T,\beta} \in \mathcal{S}(\mathbb{R})$, we can use the Cauchy-Schwarz inequality to see that for every $N = 1, 2, 3, \dots$ this is dominated by a constant depending on T , β and N times

$$\left(\sum_{j=0}^{\infty} (1 + |\lambda - \lambda_j|)^{-N} |(I - \chi_T(q, D)) \varphi_j^b(y)|^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} (1 + |\lambda - \lambda_j|)^{-N} |\varphi_j^b(y)|^2 \right)^{\frac{1}{2}}.$$

The statement then follows from (5.9) and from the local Weyl law (1.6). \square

Lemma 10 follows from (5.3), Sublemma 11, and Sublemma 13 and it completes the proof of Proposition 9. As explained at the start of the proof, the Proposition implies Theorem 1.1. \square

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208,
USA

E-mail address: csogge@math.jhu.edu

E-mail address: zelditch@math.northwestern.edu